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New solitons for the Hirota equation and generalized higher-order nonlinear Schrödinger equation with variable coefficients

Chao-Qing Dai[1,2] **and Jie-Fang Zhang**[1,3]

[1] Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua 321004, People's Republic of China
[2] Department of Information Physics, School of Sciences, Zhejiang Forestry University, Linan 311300, People's Republic of China
[3] CCAST (World Lab.), PO Box 8730, Beijing 10080, People's Republic of China

E-mail: jf_zhang@zjnu.cn

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Abstract
In this paper, multisoliton solutions of the Hirota equation with variable coefficients are obtained by the Darboux transformation based on the Ablowitz–Kaup–Newell–Segur technology. As an example, we discuss the evolutional behaviour of a two-soliton solution in a soliton control fibre system. The results reveal that one may control the interaction between the pulses by choosing the third-order dispersion parameters d_4 and h appropriately. Meanwhile, more generalized forms of bright soliton and dark soliton solutions of generalized higher order nonlinear Schrödinger equations (GHONLSE) with variable coefficients are obtained by the extended tanh-function method. Moreover, new bright and dark combined solitary wave, kink solitary wave and M-shaped solitary wave to GHONLSE with variable coefficients are firstly reported in this paper. Especially, the term proportional to α_1 resulting from the group velocity decides the group velocity and the phase shift of these new solitary waves.

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1. Introduction

During the past few decades, terrestrial and submarine communication systems have scored an incredible growth of their transmission capacity. Optical solitons have been witnessed as the objects of extensive theoretical and experimental studies due to their potential applications in long-distance communication [1] and all-optical ultrafast switching devices. The optical soliton in a dielectric fibre was first proposed by Hasegawa and Tappert [2]. The invention of high-intensity lasers helped Mollenauer *et al* [3] to verify experimentally the pioneering theoretical work on optical solitons, localized-in-time optical pulse evolution from a nonlinear change in the refractive index of the material, known as the Kerr effect, included by the light intensity distribution. It is well known that picosecond pulses are well described by the classic nonlinear Schrödinger equation (NLSE) which accounts for the group velocity dispersion (GVD) and self-phase modulation (SPM). To enlarge the information capacity, it is necessary to transmit ultrashort (subpicosecond and femtosecond) optical pulse at a high bit rate. However, when the stronger and stronger intensity of the incident light field and shorter and shorter pulses are considered, the classic NLSE fails in the physical description of the propagations of light pulses in fibres because higher order dispersion term and the higher order nonlinear effects such as the third-order dispersion (TOD), self-steepening (SS) etc cannot be neglected. The nonlinear propagation equation for femtosecond optical pulse, described by the higher order nonlinear Schrödinger equation (HONLSE), has been derived by Kodama *et al* [4] and is quite different from the well-known NLSE.

In the application of solitons, the concept of soliton control is a new and important development. Because of the exponential attenuation of the pulse along a real optical fibre, the balance between the nonlinearity and the dispersion is broken and the transmission of optical solitons is affected. In order to maintain a well-ordered transmission of solitons in a communication system, one usually introduces a slow change of the fibre parameters in the longitudinal direction or a device amplifying the insulation of heat production. So it is very significant to study NLS-type equation with variable coefficients. Picosecond soliton control, described by the variable-coefficients nonlinear Schrödinger equation (vNLSE), has been extensively studied theoretically [5–7]. More recently, studies of femtosecond soliton control, described by the variable-coefficients higher order nonlinear Schrödinger equation (vHONLSE), have been developed [8–13]. The dynamical evolution of transmission of femtosecond solitons is modelled according to the following form of a spacetime-dependent vHONLSE [12]:

$$u_z + \alpha_1(z)u_t + \alpha_2(z)u + \mathrm{i}\alpha_3(z)u_{tt} + \alpha_4(z)u_{ttt} + \mathrm{i}\alpha_5(z)|u|^2u \\ + \alpha_6(z)(|u|^2u)_t + \alpha_7(z)u(|u|^2)_t = 0. \qquad (1)$$

In equation (1), $u(z, t)$ is the complex envelope of the electrical field, $\alpha_1 \sim \alpha_7$ are all the functions of the arguments z. z and t are the propagation distance and time, respectively. And the term proportional to α_1 results from the group velocity; α_2 is the heating-insulating amplification or loss; α_3 and α_4 represent GVD and TOD, respectively; the term proportional to α_5 represents the SPM; the term proportional to α_6 results from including the first derivative of the slowly varying part of the nonlinear polarization, and it is responsible for SS and shock formation at a pulse edge; the last term proportional to α_7 represents the delayed nonlinear response effect. This equation can also describe femtosecond optical pulse propagation in inhomogeneous fibres [13], which is not discussed widely, to our knowledge. From equation (1), we can see that the physical quantity u is not defined in the frame of the group velocity owing to the existence of the term $\alpha_1(z)$. If the coefficients $\alpha_1(z) \sim \alpha_7(z)$ in the present paper are all constants, then equation (1) can be transformed into the equation defined in the frame of the group velocity in [14] by means of the transformation $t = t - \alpha_1 z$. However, we have not found similar transformation to change equation (1) into the equation in [8, 9, 11] due to these variable coefficients $\alpha_1(z) \sim \alpha_7(z)$.

Mathematically speaking, the Hirota equation represents the integrable version of the NLSE. Hirota [15] had already obtained multisoliton solutions of the Hirota equation using the inverse scattering transform. The Hirota equation has been investigated in the form of soliton solution with vanishing boundary conditions by several authors [15–17]. Soliton solution with non-vanishing boundary conditions of this equation has been discussed in [18, 19]. But in the systems considered so far, the case of variable coefficients is less. The Painlevé analysis of the Hirota equation with variable coefficients is given in [20].

We know there are some wealth methods for finding special solutions of a constant coefficient partial differential nonlinear equation such as the inverse scattering transformation (IST), bilinear method, symmetry reductions, Bäcklund and Darboux transformations and so on. Because it is very difficult to solve variable-coefficient nonlinear equations, they were often studied by means of numerical analysis or approximate methods. In this paper, multisoliton solutions of the Hirota equation are obtained by the Darboux transformation based on the Ablowitz–Kaup–Newell–Segur (AKNS) technology in section 2. However, to our knowledge, dark soliton cannot be derived by the Darboux transformation based on the AKNS technology. To derive dark soliton solution, we use the extended tanh-function method [11], by means of which bright soliton can be obtained at one time in section 3. The bright soliton solution has a more generalized form than the one obtained by the Darboux transformation based on the AKNS technology, and the dark soliton solution has also a more generalized form than the one in [13]. Searching for new types of solitary wave is of particular interest because of their potential application in telecommunication and ultrafast signal routing systems. In section 4, we derive W-shaped solitary wave solutions by assuming real amplitude instead of assuming imaginary amplitude in [13, 16]. Moreover, we present firstly some new types of solitary wave solutions, including bright and dark combined solitary wave, kink solitary wave and M-shaped solitary wave, to the generalized higher order nonlinear Schrödinger equation with variable coefficients.

2. Multisoliton solutions of the Hirota equation with variable coefficients

We now turn to obtain multisoliton solutions by the Darboux transformation based on the AKNS technology [21]. Generally, equation (1) is not integrable. To solve equation (1), we consider the following Hirota condition:

$$\alpha_2 = \frac{\alpha_5\alpha_{3z} - \alpha_3\alpha_{5z}}{2\alpha_3\alpha_5}, \qquad 3\alpha_4\alpha_5 = (3\alpha_6 + 2\alpha_7)\alpha_3, \qquad \alpha_6 + \alpha_7 = 0; \tag{2}$$

then equation (1) is reduced to the Hirota equation with variable coefficients:

$$u_z + \alpha_1(z)u_t + \alpha_2(z)u + \mathrm{i}\alpha_3(z)u_{tt} + \alpha_4(z)u_{ttt} + \mathrm{i}\alpha_5(z)|u|^2u + \alpha_6(z)|u|^2u_t = 0. \tag{3}$$

By employing the AKNS technology one can construct the linear eigenvalue problem for Hirota equation (3) with condition (2) as follows:

$$\begin{aligned} \Phi_t &= L\Phi = \begin{pmatrix} \lambda & \sqrt{\frac{\alpha_5}{2\alpha_3}}u \\ -\sqrt{\frac{\alpha_5}{2\alpha_3}}u^* & -\lambda \end{pmatrix}\Phi, \\ \Phi_z &= M\Phi = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\Phi, \end{aligned} \tag{4}$$

with

$$\begin{aligned} A &= -\left[4\alpha_4\lambda^3 + 2\mathrm{i}\alpha_3\lambda^2 + \alpha_1\lambda + \frac{\alpha_4\alpha_5}{\alpha_3}|u|^2\lambda + \frac{\alpha_4\alpha_5}{2\alpha_3}(u^*u_t - uu_t^*) + \mathrm{i}\frac{\alpha_5}{2}|u|^2\right] \\ B &= -\sqrt{\frac{\alpha_5}{2\alpha_3}}\left[4\alpha_4u\lambda^2 + 2(\alpha_4u_t + \mathrm{i}\alpha_3u)\lambda + \alpha_4\left(u_{tt} + \frac{\alpha_5}{\alpha_3}u|u|^2\right) + \mathrm{i}\alpha_3u_t + \alpha_1u\right] \\ C &= \sqrt{\frac{\alpha_5}{2\alpha_3}}\left[4\alpha_4u^*\lambda^2 - 2(\alpha_4u_t^* - \mathrm{i}\alpha_3u^*)\lambda + \alpha_4\left(u_{tt}^* + \frac{\alpha_5}{\alpha_3}u^*|u|^2\right) - \mathrm{i}\alpha_3u_t^* + \alpha_1u^*\right], \end{aligned} \tag{5}$$

where $*$ represents the complex conjugate. It is easy to verify that equation (3) can be recovered from the compatibility condition $L_z - M_t + [L, M] = 0$.

Next we turn to solve Hirota equation (3) with condition (2) by employing the Darboux transformation. Firstly, introducing transformation

$$\Phi[1] = (\lambda I - S)\Phi, \qquad S = H\Lambda H^{-1}, \qquad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2), \tag{6}$$

where H and I are nonsingular and identity matrices, respectively. Thus requiring

$$\Phi_t[1] = L_1\Phi[1], \qquad L_1 = \lambda\sigma_3 + P_1 \tag{7}$$

with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad P = \sqrt{\frac{\alpha_5}{2\alpha_3}}\begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}, \tag{8}$$

we obtain the Darboux transformation for Hirota equation (3) with condition (2) in the form

$$P_1 = P + [\sigma_3, S]. \tag{9}$$

It is easy to verify that, if $(\varphi_1, \varphi_2)^T$ is a solution of equation (4) corresponding to eigenvalue λ_1, then $(-\varphi_2^*, \varphi_1^*)^T$ is also a solution of equation (4) corresponding to the eigenvalue $-\lambda_1^*$. Thus if taking

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1^* \end{pmatrix}, \qquad H = \begin{pmatrix} \varphi_1 & -\varphi_2^* \\ \varphi_2 & \varphi_1^* \end{pmatrix}, \tag{10}$$

we can obtain from equation (6)

$$S_{kl} = -\lambda_1^*\delta_{kl} + \frac{(\lambda_1 + \lambda_1^*)\varphi_k\varphi_l^*}{\Delta}, \tag{11}$$

where $k, l = 1, 2$, and $\Delta = \det H = |\varphi_1|^2 + |\varphi_2|^2 \neq 0$. And from equation (9) we obtain the fundamental Darboux transformation as follows:

$$u_1 = u + 2\sqrt{\frac{2\alpha_3}{\alpha_5}}S_{12} = u + 2\sqrt{\frac{2\alpha_3}{\alpha_5}}\frac{(\lambda_1 + \lambda_1^*)\varphi_1\varphi_2^*}{|\varphi_1|^2 + |\varphi_2|^2}. \tag{12}$$

Analogous to this procedure and taking the Darboux transformation n times, we find the following formula:

$$u_n = u + 2\sqrt{\frac{2\alpha_3}{\alpha_5}}\sum_{m=1}^{n}\frac{(\lambda_m + \lambda_m^*)\varphi_{1,m}(\lambda_m)\varphi_{2,m}^*(\lambda_m)}{A_m}, \tag{13}$$

with

$$\begin{aligned} \varphi_{k,m+1}(\lambda_{m+1}) &= (\lambda_{m+1} + \lambda_m^*)\varphi_{k,m}(\lambda_{m+1}) - \frac{B_m}{A_m}(\lambda_m + \lambda_m^*)\varphi_{k,m}(\lambda_m), \\ A_m &= |\varphi_{1,m}(\lambda_m)|^2 + |\varphi_{2,m}(\lambda_m)|^2, \\ B_m &= \varphi_{1,m}(\lambda_{m+1})\varphi_{1,m}^*(\lambda_m) + \varphi_{2,m}(\lambda_{m+1})\varphi_{2,m}^*(\lambda_m), \end{aligned} \tag{14}$$

where $m = 1, \ldots, n$, $k = 1, 2$ and $(\varphi_{1,1}(\lambda_1), \varphi_{2,1}(\lambda_1))^T$ is the eigenfunction of equation (4) corresponding to λ_1 for u. Inserting the zero solution of equation (3) as $u = 0$ into equation (12), one can obtain the one soliton solution for equation (3). Using that one soliton solution as the seed solution in equation (12), we can derive the two-soliton solution. Thus in recursion, one can generate up to n-soliton solution. Here we give one- and two-soliton solutions in explicit forms.

Substituting the complex spectral parameter $\lambda_1 = \frac{1}{2}(\eta_1 + \mathrm{i}\xi_1)$ into Lax pair (4), we can derive

$$\varphi_1 = \exp\left(\frac{\theta_1}{2} + \mathrm{i}\frac{\phi_1}{2}\right), \qquad \varphi_2 = \exp\left[-\left(\frac{\theta_1}{2} + \mathrm{i}\frac{\phi_1}{2}\right)\right], \tag{15}$$

in equation (11) with $u = 0$, where

$$\begin{aligned} \theta_1 &= \eta_1\left[t - \int_0^z \alpha_1(z)\,\mathrm{d}z - \left(\eta_1^2 - 3\xi_1^2\right)\int_0^z \alpha_4(z)\,\mathrm{d}z - 2\xi_1\int_0^z \alpha_3(z)\,\mathrm{d}z\right] - \theta_{10}, \\ \phi_1 &= \xi_1\left(t - \int_0^z \alpha_1(z)\,\mathrm{d}z\right) - \xi_1\left(3\eta_1^2 - \xi_1^2\right)\int_0^z \alpha_4(z)\,\mathrm{d}z - \left(\eta_1^2 - \xi_1^2\right)\int_0^z \alpha_3(z)\,\mathrm{d}z - \phi_{10}, \end{aligned} \tag{16}$$

with θ_{10}, ϕ_{10} being arbitrary real constants and they represent the initial position and the initial phase of soliton, respectively. Thus the one-soliton is of the form

$$u_1 = \eta_1\sqrt{\frac{2\alpha_3}{\alpha_5}}\,\mathrm{sech}\,\theta_1\exp(\mathrm{i}\phi_1), \tag{17}$$

where θ_1, ϕ_1 satisfy equation (16). Due to the more generalized bright soliton solution obtained in section 3, we intend to present the properties of bright soliton in the following discussion.

When $n = 2$, from equation (13), we obtain the two-soliton solution as follows:

$$u_2 = \sqrt{\frac{2\alpha_3}{\alpha_5}}\frac{G}{F}, \tag{18}$$

where

$$\begin{aligned} G &= a_1\cosh\theta_2\,\mathrm{e}^{\mathrm{i}\phi_1} + a_2\cosh\theta_1\,\mathrm{e}^{\mathrm{i}\phi_2} + \mathrm{i}a_3(\sinh\theta_2\,\mathrm{e}^{\mathrm{i}\phi_1} - \sinh\theta_1\,\mathrm{e}^{\mathrm{i}\phi_2}), \\ F &= b_1\cosh(\theta_1 + \theta_2) + b_2\cosh(\theta_1 - \theta_2) + b_3\cos(\phi_2 - \phi_1) \end{aligned} \tag{19}$$

with

$$\begin{aligned} \theta_k &= \eta_k\left[t - \int_0^z \alpha_1(z)\,\mathrm{d}z - \left(\eta_k^2 - 3\xi_k^2\right)\int_0^z \alpha_4(z)\,\mathrm{d}z - 2\xi_k\int_0^z \alpha_3(z)\,\mathrm{d}z\right] - \theta_{k0}, \\ \phi_k &= \xi_k\left(t - \int_0^z \alpha_1(z)\,\mathrm{d}z\right) - \xi_k\left(3\eta_k^2 - \xi_k^2\right)\int_0^z \alpha_4(z)\,\mathrm{d}z - \left(\eta_k^2 - \xi_k^2\right)\int_0^z \alpha_3(z)\,\mathrm{d}z - \phi_{k0}, \end{aligned} \tag{20}$$

and

$$\begin{aligned} a_k &= \frac{\eta_k}{2}\left[\eta_k^2 - \eta_{3-k}^2 + (\xi_1 - \xi_2)^2\right], \\ b_k &= \tfrac{1}{4}\{[\eta_1 + (-1)^k\eta_2]^2 + (\xi_1 - \xi_2)^2\}, \\ a_3 &= \eta_1\eta_2(\xi_1 - \xi_2), \qquad b_3 = -\eta_1\eta_2, \end{aligned} \tag{21}$$

and θ_{k0} and ϕ_{k0} are two arbitrary real constants, and we have utilized $\lambda_k = \frac{1}{2}(\eta_k + \mathrm{i}\xi_k)$, $(k = 1, 2)$. Based on the exact solution (18), we can discuss the transmission properties of two femtosecond optical solitons in optical fibre systems. From the expression of θ_k, one can distinctly find that the width of each soliton in the two-soliton solution (18) is determined by the real part of spectral parameter η_k, and the velocity of each soliton is related to $\alpha_1(z) + \left(\eta_k^2 - 3\xi_k^2\right)\alpha_4(z) + 2\xi_k\alpha_3(z)$, which is dependent not only on the real part η_k and the imaginary part ξ_k of spectral parameters , but also on the distributed parameters $\alpha_1(z), \alpha_4(z), \alpha_3(z)$. Thus, we can trap the velocity of each soliton to form a bound-soliton solution by devising the distributed parameters $\alpha_1(z), \alpha_4(z), \alpha_3(z)$.

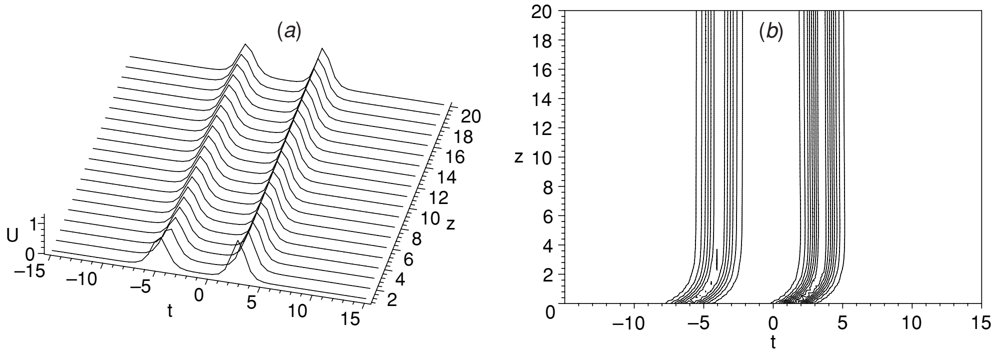

Figure 1. (*a*) An evolution plot of the intensity of two solitons with the parameters $d_2 = g = r = \sigma = -d_1 = -1, d_5 = -2, d_4 = h = -0.1, \eta_1 = -1, \eta_2 = \theta_{10} = \theta_{20} = 1.15, \xi_1 = \xi_2 = 0.6, \phi_{10} = \phi_{20} = 0$; (*b*) the corresponding contour plot.

Now we discuss the evolutional behaviour of two-soliton solution (18) in the following soliton control system similarly to the system in [9]. The parameter results from the group velocity $\alpha_1(z)$,

$$\alpha_1(z) = d_1 \exp(-rz), \tag{22}$$

the group velocity dispersion parameter $\alpha_3(z)$,

$$\alpha_3(z) = d_3 \exp(-gz), \tag{23}$$

the third-order dispersion parameter $\alpha_4(z)$,

$$\alpha_4(z) = d_4 \exp(-hz), \tag{24}$$

and the nonlinearity parameter $\alpha_5(z)$,

$$\alpha_5(z) = d_5 \exp(-\sigma z), \tag{25}$$

where d_1 and r, d_3 and g, d_4 and h, d_5 and σ are the parameters to describe the group velocity, the group velocity dispersion, third-order dispersion and the nonlinearity, respectively. When g and h are both positive, this system represents the dispersion decreasing optical fibre system. From the Hirota condition (2), the gain/loss distributed function is of the form $\alpha_2(z) = (\sigma - g)/2$ ($\sigma > g$ for the gain; $\sigma < g$ for the loss). From the system (22)–(25), we can see that there are too many control parameters to clearly analyse the propagation behaviours of two-soliton. Therefore, here we only consider the different evolutional behaviours resulting from the change of the third-order dispersion parameters d_4 and h, which is different from the case in [9]. From figures 1–3, we can see that with suitable initial separation, two solitons do not interact for $d_4 h > 0$, while they elastically collide for $d_4 < 0, h > 0$. Concretely speaking, when $d_4 < 0, h < 0$, two solitons independently propagate with invariant separation. When $d_4 > 0, h > 0$, they propagate along fibre with increasing separation. It seems that there is a repulsive force between two solitons. The results show that one may control the interaction between the pulses by choosing the parameters d_4 and h appropriately.

3. Bright and dark soliton solutions

From the procedure of the Darboux transformation based on the AKNS technology, we can clearly see that it is difficult to obtain dark soliton solution by means of this method. However,

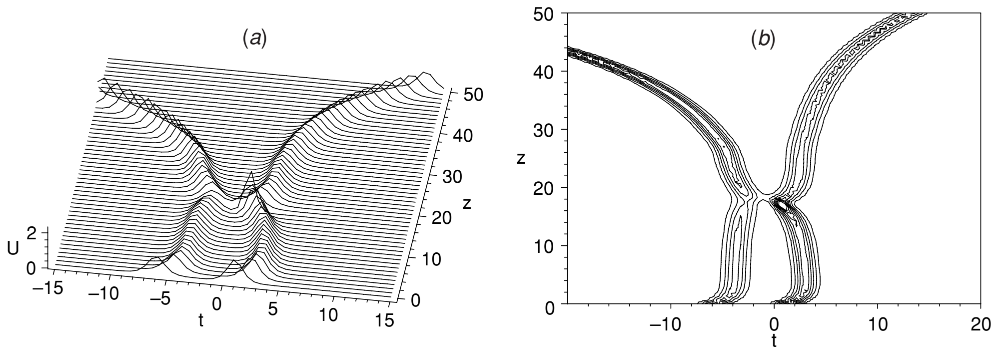

Figure 2. (*a*) An evolution plot of the intensity of two solitons with the parameters $d_4 = -h = -0.1$, the other parameters being the same as in figure 1; (*b*) the corresponding contour plot.

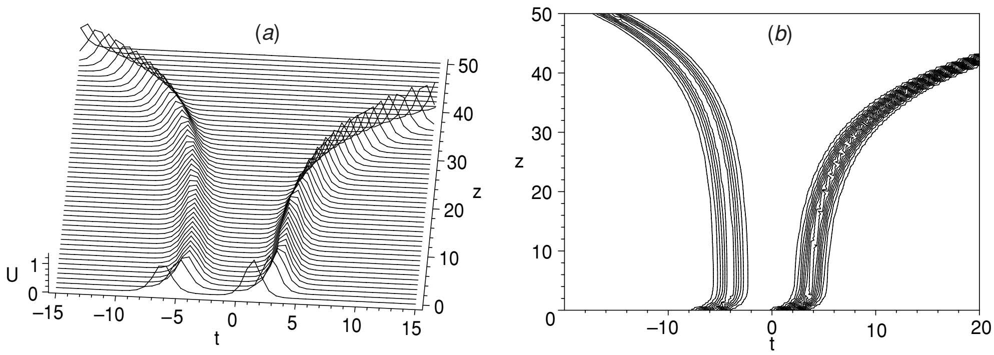

Figure 3. (*a*) An evolution plot of the intensity of two solitons with the parameters $d_4 = h = 0.1$, the other parameters being the same as in figure 1; (*b*) the corresponding contour plot.

some distinct features make dark solitons very physically interesting. For example, Zhao and Bourkoff [22] compared bright and dark pulse properties showing that the dark pulse spreads more slowly with loss and is less sensitive to noise than a bright one. To derive dark soliton solution, we use the extended tanh-function method [11], by means of which bright soliton can be obtained at one time.

Firstly, we make the ansatz

$$u(z,t) = v(z,t)\exp(\mathrm{i}\varphi(z,t)), \tag{26}$$

where the envelope function $v(z,t)$ is real and $\varphi(z,t)$ is phase function, then substituting equation (26) into equation (1) and separating the real and imaginary parts of the resulting equation, and using the standard leading order analysis, we have $v(z,t) = a_0(z) + a_1(z)\phi(\xi)$, $\xi(z,t) = p(z)t + q(z)$, $\varphi(z,t) = \Gamma(z)t + \Omega(z)$. And with the help of auxiliary equation

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 = c_0 + c_2\phi^2 + c_4\phi^4, \tag{27}$$

we can derive bright soliton and dark soliton solutions.

When $c_4 < 0$, $c_2 > 0$, bright soliton solution

$$u(z,t) = A_1\sqrt{\frac{-c_2}{c_4}}\exp\left(-\int_0^z \alpha_2(z)\,\mathrm{d}z\right)\mathrm{sech}[\sqrt{c_2}\theta(z,t)]\exp[\mathrm{i}\varphi(z,t)]. \tag{28}$$

When $c_4 > 0, c_2 < 0$, dark soliton solution

$$u(z,t) = A_1\sqrt{\frac{-c_2}{2c_4}}\exp\left(-\int_0^z \alpha_2(z)\,\mathrm{d}z\right)\tanh\left[\sqrt{-\frac{c_2}{2}}\theta(z,t)\right]\exp[\mathrm{i}\varphi(z,t)], \tag{29}$$

where

$$\begin{aligned}\theta(z,t) &= A_2t - A_2\int_0^z \alpha_1(z)\,\mathrm{d}z + 2A_2A_3\int_0^z \alpha_3(z)\,\mathrm{d}z + B_3\int_0^z \alpha_4(z)\,\mathrm{d}z + A_5,\\ \varphi(z,t) &= A_3t - A_3\int_0^z \alpha_1(z)\,\mathrm{d}z + B_1\int_0^z \alpha_3(z)\,\mathrm{d}z + B_2\int_0^z \alpha_4(z)\,\mathrm{d}z + A_4,\end{aligned} \tag{30}$$

and

$$B_1 = A_3^2 - A_2^2c_2, \qquad B_2 = A_3^3 - 3A_3A_2^2c_2, \qquad B_3 = 3A_2A_3^2 - c_2A_2^3, \tag{31}$$

with certain parametric conditions

$$\alpha_4(z) = -\frac{A_1^2}{6A_2^2c_4}[3\alpha_6(z) + 2\alpha_7(z)]\exp\left(-2\int_0^z \alpha_2(z)\,\mathrm{d}z\right), \tag{32}$$

$$[3\alpha_6(z) + 2\alpha_7(z)]\alpha_3(z) = 3[\alpha_5(z) - 2\alpha_6(z)A_3 - 2\alpha_7(z)A_3]\alpha_4(z), \tag{33}$$

where $A_1 \sim A_5$ are arbitrary constants. From equations (28) and (29), one can see that the velocity of the bright soliton and dark soliton solutions is related to $A_2\alpha_1(z,t) - 2A_2A_3\alpha_3(z,t) - B_3\alpha_4(z,t)$, the time shift is determined by $A_2\int_0^z \alpha_1(z)\,\mathrm{d}z - 2A_2A_3\int_0^z \alpha_3(z)\,\mathrm{d}z - B_3\int_0^z \alpha_4(z)\,\mathrm{d}z - A_5$, and the phase shift depends on $A_3\alpha_1(z) - B_1\alpha_3(z) - B_2\alpha_4(z)$. Since the group velocity and the time shift of the solitons are dependent on z, the centre position of the solitons changes along the propagation direction of the fibre, which means that one may design a fibre system to control the velocity, time shift and phase shift of solitons by choosing appropriate optical fibre parameters $\alpha_1(z), \alpha_3(z)$ and $\alpha_4(z)$. In addition, from equation (32), the gain or loss coefficient $\alpha_2(z)$ is determined by the parameters $\alpha_4(z), \alpha_6(z), \alpha_7(z)$. Actually, the bright soliton solution (28) includes the one-soliton solution (17) because the parametric conditions (32) and (33) comprise the integrable condition (2). Due to similar reason, the dark soliton solution (29) has a more generalized form than the one in [8].

Now we discuss the features of the bright soliton solution (28) for a periodic distributed amplification system [11, 23]. We suppose the parameters $\alpha_1(z)$ and $\alpha_3(z)$

$$\alpha_1(z) = d_1\sin(z), \qquad \alpha_3(z) = d_2\exp(-d_3z), \tag{34}$$

the parameters $\alpha_6(z)$ and $\alpha_7(z)$

$$\alpha_6(z) = -d_4\sin(z), \qquad \alpha_7(z) = -d_5\cos(z), \tag{35}$$

the TOD parameter $\alpha_4(z)$

$$\alpha_4(z) = \frac{d_6}{6}(3d_4\sin(z) + 2d_5\cos(z))\exp(\sigma z), \tag{36}$$

where d_1 depends on the group velocity, d_2 and d_3 are related to GVD, d_4 and d_5 describe SS and delayed nonlinear response effect, respectively. And $d_6 = \frac{A_1^2}{A_2^2c_4}$, which is related to TOD. Then the nonlinearity parameter $\alpha_5(z)$,

$$\alpha_5(z) = -\frac{2d_2}{d_6}\exp(-d_3z - \sigma z) - 2A_3(d_4\sin(z) + d_5\cos(z)), \tag{37}$$

and the gain/loss function determined by equation (32) is $\alpha_2(z) = -\frac{\sigma}{2}$ ($\sigma > 0$ for the gain and $\sigma < 0$ for the loss). Moreover, the velocity of the bright soliton depends on $A_2d_1\sin(z) - \frac{B_3d_6}{6}(3d_4\sin(z) + 2d_5\cos(z))\exp(\sigma z) - 2A_2A_3d_2\exp(-d_3z)$, the phase shift is

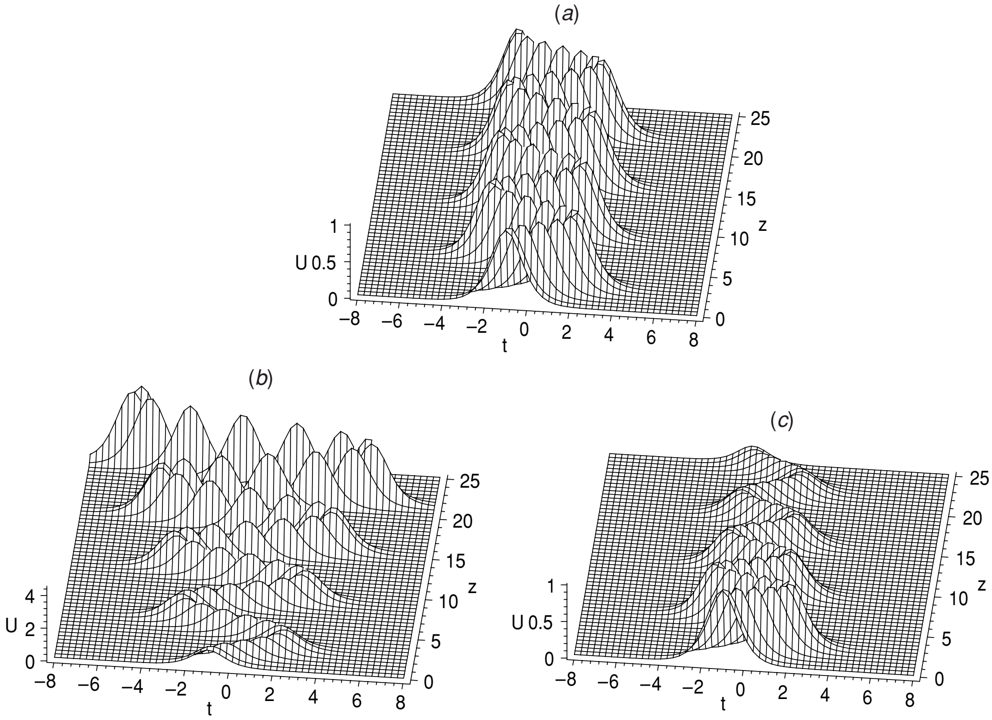

Figure 4. The intensity $U = |u|^2$ of bright soliton (27): (a) $\sigma = 0$; (b) $\sigma = 0.06$ (gain); (c) $\sigma = -0.06$ (loss).

related to $A_3 d_1 \sin(z) - \frac{B_2 d_6}{6}(3d_4 \sin(z) + 2d_5 \cos(z)) \exp(\sigma z) - B_1 d_2 \exp(-d_3 z)$, the time shift is of the form

$$\frac{2A_2 A_3 d_2}{d_3} \exp(-d_3 z) - A_2 d_1 \cos(z) - \frac{d_6 B_3}{6(\sigma^2 + 1)}[(3d_4\sigma + 2d_5)\sin(z) + (2d_5\sigma - 3d_4)\cos(z)] \exp(\sigma z).$$

Besides, the pulse width and the wave number of the bright soliton is $\frac{1}{\sqrt{c_2 A_2}}$ and A_3, which remain constant during propagation along the fibre.

Here we take the parameters $d_1 = c_4 = -1, d_2 = d_4 = d_5 = d_6 = 1, d_3 = 0.5, A_1 = A_2 = A_3 = c_2 = 1, A_4 = A_5 = 0$. The nonlinear evolution behaviour of the bright soliton is presented in figure 4. From figure 4 one can clearly see that the group velocity and time shift of the soliton vary with the dispersion distribution, and the intensity of the soliton increases or decreases depending on the sign of σ, while the soliton pulse width remains a constant. This is one of the important properties of soliton.

Next, we also discuss the evolution behaviour of the dark soliton (29) in the following soliton control system:

the parameters $\alpha_1(z)$ and $\alpha_3(z)$,

$$\alpha_1(z) = d_1 \exp(-kz), \qquad \alpha_3(z) = d_3 \exp(-gz), \tag{38}$$

the parameters $\alpha_6(z)$ and $\alpha_7(z)$,

$$\alpha_6(z) = d_6 \exp(\sigma z - hz), \qquad \alpha_7(z) = d_7 \exp(\sigma z - hz), \tag{39}$$

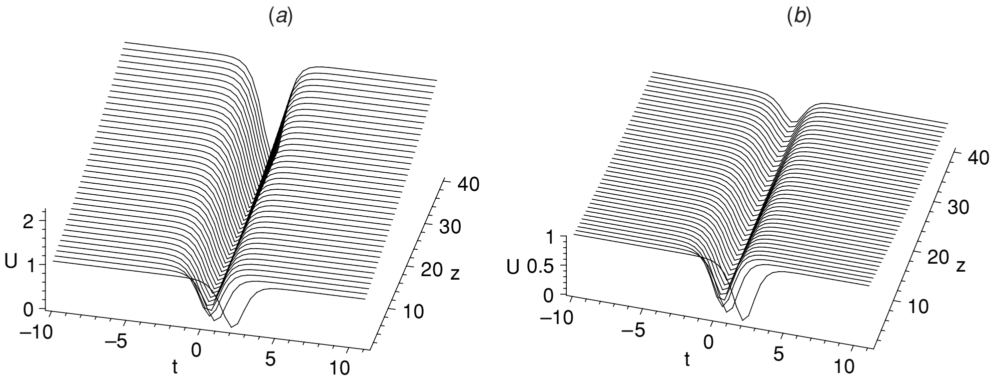

Figure 5. The evolution of dark soliton (28) $U = |u|^2$: (*a*) $\sigma = 0.01$ (gain); (*b*) $\sigma = -0.01$ (loss).

the TOD parameter $\alpha_4(z)$,

$$\alpha_4(z) = d_4 \exp(-hz), \tag{40}$$

where d_1 depends on the group velocity, d_3 and g are related to GVD, d_4 and h are related to TOD($g > 0, h > 0$ for dispersion decreasing fibres), d_6 and d_7 describe SS and delayed nonlinear response effect, respectively, and $3d_6 + 2d_7 = 1$. From equation (32), the gain/loss function $\alpha_2(z) = -\frac{\sigma}{2}$. When $\sigma > 0$, solution (29) can be applied to the problem of dark soliton propagating in optical fibre amplifiers. In this case, the dark soliton is amplified as shown in figure 5(*a*), where $d_1 = d_4 = h = 1, k = 0.5, g = 0.6, A_1 = A_2 = A_3 = -c_2 = c_4 = 1, A_4 = A_5 = 0$. When $\sigma < 0$, solution (29) has application to the soliton management communication links where fibre losses are compensated periodically by an amplification system. This case is shown in figure 5(*b*). From these plots it can be seen that the dark soliton keeps its shape even if the amplitude and velocity are changing in soliton management system, which is similar to bright soliton as mentioned.

4. New types of solitary wave solutions

So far, we have obtained the one- and two-soliton solutions, and the dark soliton solution. Now a significant and interesting issue is whether there are other combined solitary waves besides the W-shaped solitary wave in [13]. We could not yet obtain the Lax pair of equation (1) for combined solitary waves in spite of making great efforts to search for it. Fortunately, we can obtain some new types of solitary wave solutions by the direct method. It is worthwhile to note that we here derive these solitary wave solutions, including bright and dark combined solitary wave, W-shaped solitary wave and M-shaped solitary wave, by assuming real amplitude instead of assuming imaginary amplitude in [13, 16].

4.1. Bright and dark combined solitary wave and kink solitary wave solutions

Substituting equation (26) with the real amplitude

$$\begin{aligned} v(z,t) &= a_0(z) + a_1(z)\,\mathrm{sech}(\xi) + b_1(z)\tanh(\xi), \\ \xi(z,t) &= p(z)t + q(z), \varphi(z,t) = \Gamma(z)t + \Omega(z), \end{aligned} \tag{41}$$

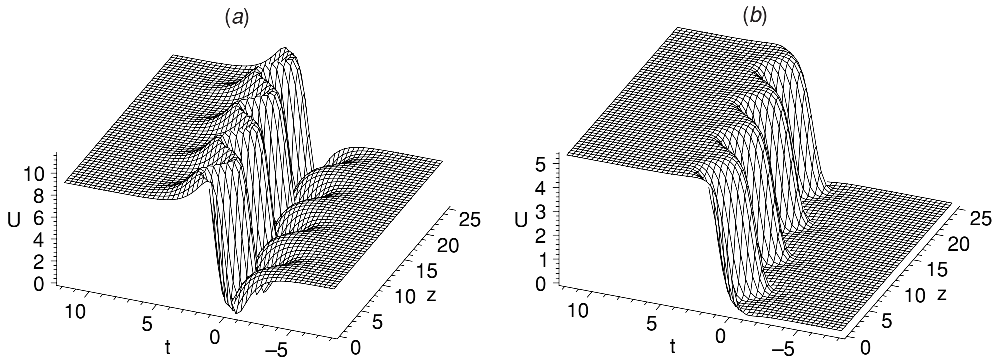

Figure 6. (*a*) Intensity of bright and dark combined solitary wave $U = |u|^2$ with the parameters $A_1 = 0.5$, $A_2 = 1.5$, $A_3 = 2.5$, $A_4 = 1$, $\sigma = 0$; (*b*) Intensity of kink solitary wave $U = |u|^2$ with the parameters $A_1 = A_4 = 1$, $A_2 = 0.2$, $A_3 = 1.3$, $\sigma = 0$.

into equation (1), we derive a new bright and dark combined solitary wave. It is in the form of

$$u(z,t) = \exp\left[-\int_0^z \alpha_2(z)\,\mathrm{d}z\right]\{A_1 + A_2\,\mathrm{sech}[\xi(z,t)] + A_3\tanh[\xi(z,t)]\}\exp[\mathrm{i}\varphi(z,t)], \tag{42}$$

where

$$\xi(z,t) = A_4\left(t - \int_0^z \alpha_1(z)\,\mathrm{d}z\right) + A_6,\ \varphi(z,t) = A_5\left(t - \int_0^z \alpha_1(z)\,\mathrm{d}z\right) + A_7,\ A_5 = -\frac{\alpha_5(z)}{\alpha_6(z)},$$

with certain parametric conditions

$$\alpha_3(z) = \alpha_4(z) = 0, \qquad 3\alpha_6(z) + 2\alpha_7(z) = 0, \tag{43}$$

which for the solution (42) is the same as the limit conditions for W-shaped solitary wave in [20], and $A_1 \sim A_7$ are real constants.

From the solitary wave solution (42), one knows that the group velocity and the phase shift are decided by $\alpha_1(z)$, the amplitude of the solitary wave depends on the gain or loss coefficient $\alpha_2(z)(\alpha_2(z) < 0$ for the gain and $\alpha_2(z) > 0$ for the loss), and the pulse width of the solitary wave remains constant $\frac{1}{A_3}$. This type of combined solitary wave can describe the properties of both bright and dark solitary waves, and their amplitudes do not approach zero when the time variable approaches infinity unless it changes into the kink solitary wave. This solitary wave depends on its coefficient A_2 and A_3 relations. When the absolute of the coefficients A_2 and A_3 gradually decrease, the interaction between the bright soliton and the dark soliton gets bit by bit strong, and the joint part of the bright soliton and the dark soliton becomes little by little smooth. Due to the mutual modulation of the bright soliton and the dark soliton, this type of combined solitary wave turns into kink solitary wave. These properties can be seen from figure 6, where the parameters are selected as $\alpha_1(z) = [0.1 + \cos(z)]\exp(\sigma z)$, $\alpha_2(z) = -\sigma/2$, $\sigma = 0$. To our knowledge, bright and dark combined solitary wave and kink solitary wave to this generalized higher order nonlinear Schrödinger equation with variable coefficients are firstly reported in this paper.

4.2. W-shaped solitary wave

If $A_3 = 0$ and $A_1A_2 < 0$, $|A_2| > |A_1|$, W-shaped solitary wave can be obtained. Compared to the one in [13], this solution add the control parameter $\alpha_1(z)$, which determines the group velocity and the phase shift of this solitary wave. Like bright and dark combined solitary

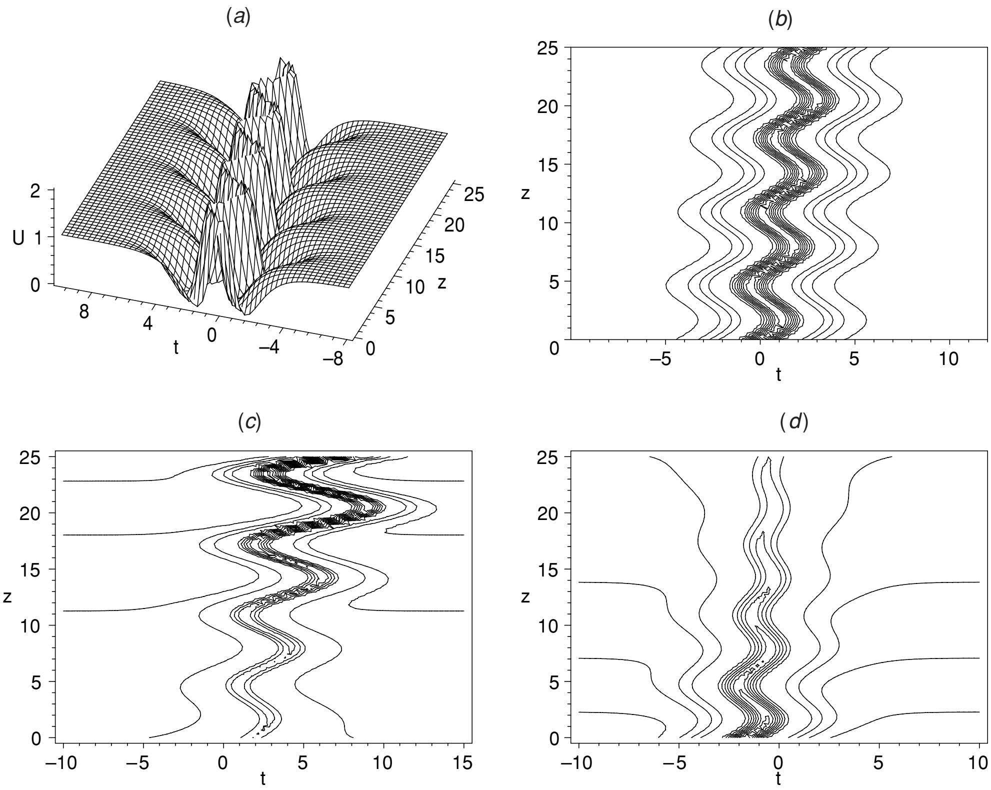

Figure 7. (a) Intensity of W-shaped solitary wave $U = |u|^2$ with the parameters $A_1 = A_3 = 1$, $A_2 = 2.414$, $A_5 = 0, \sigma = 0$. (b)–(d) The corresponding contour plot: (b) $\sigma = 0$, (c) $\sigma = 0.06$ (gain), (d)$\sigma = -0.06$ (loss).

wave, this type of combined solitary wave shown in figure 7 can also describe the properties of both bright and dark solitary waves, and their amplitudes do not approach zero when the time varies. In figure 7, the parameters $\alpha_1(z)$ and $\alpha_2(z)$ have the same selections as in figure 6. The propagation of this solitary wave in gain and loss media is shown in figures 7(c) and (d).

4.3. M-shaped solitary wave

Inserting equation (26) with the real amplitude

$$\begin{aligned} v(z,t) &= a_0(z) + a_1(z)\tanh(\xi)\,\mathrm{sech}(\zeta), \qquad & \xi(z,t) &= p(z)(t+q(z)), \\ \zeta(z,t) &= f(z)(t+q(z)), \qquad & \varphi(z,t) &= \Gamma(z)t + \Omega(z), \end{aligned} \tag{44}$$

into equation (1), with these parametric conditions (43), we can obtain M-shaped solitary wave

$$u(z,t) = \left\{A_1 \exp\left[-\int_0^z \alpha_2(z)\,\mathrm{d}z\right]\tanh[A_2\theta(z,t)]\,\mathrm{sech}[A_3\theta(z,t)]\right\}\exp[\mathrm{i}\varphi(z,t)], \tag{45}$$

where

$$\theta(z,t) = t - \int_0^z \alpha_1(z)\,\mathrm{d}z + A_5,\ \varphi(z,t) = A_4\left(t - \int_0^z \alpha_1(z)\,\mathrm{d}z\right) + A_6,\ A_4 = -\frac{\alpha_5(z)}{\alpha_6(z)},$$

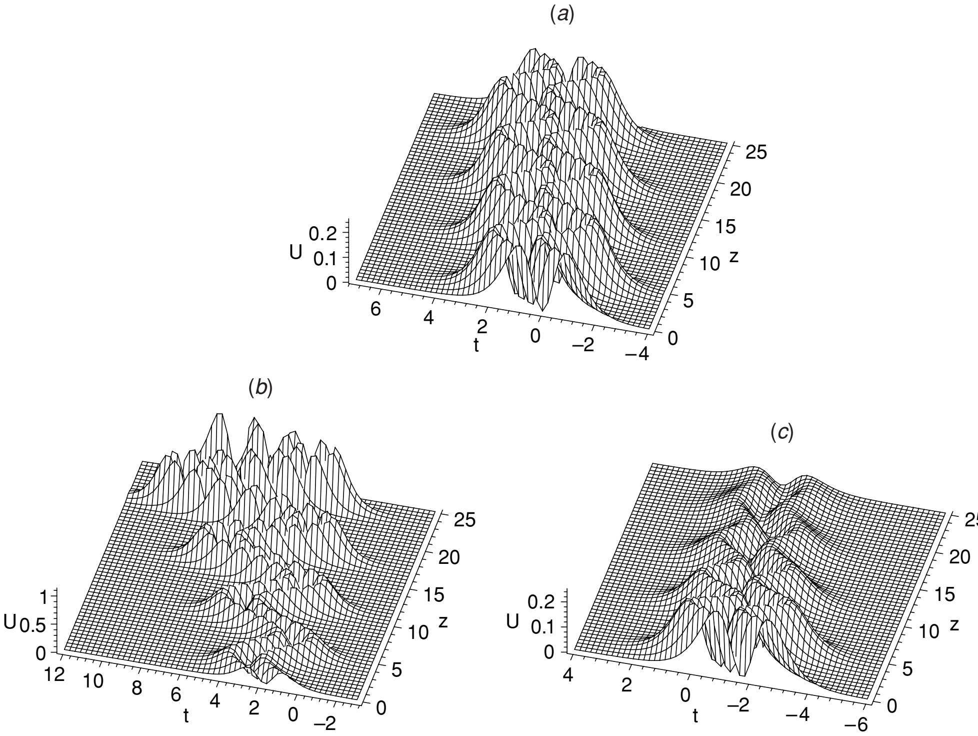

Figure 8. Intensity of M-shaped solitary wave (44) $U = |u|^2$ (*a*) $\sigma = 0$; (*b*) $\sigma = 0.06$ (gain); (*c*) $\sigma = -0.06$ (loss).

and $A_1 \sim A_6$ are real constants. From the solution (45), one can see that the solitary wave is composed of the product of a bright solitary wave and a dark solitary wave. Therefore, it can be called an inter-modulated solitary wave. It is also named after M-shaped solitary wave due to its figuration shown in figure 8. This solitary wave might be one of the possible models for femtosecond dark pulse transmission with finite width background bright pulse. The propagation of this solitary wave in gain and loss media is shown in figures 8(*b*) and (*c*) with the same selections of the parameters $\alpha_1(z)$ and $\alpha_2(z)$ as in figure 6. To our knowledge, this solitary wave to this generalized higher order nonlinear Schrödinger equation with variable coefficients is firstly presented in this paper.

From the parametric conditions (43), one can see that both the GVD and TOD must be compensated to form these new combined solitary waves. Generally, the simultaneous compensation of both GVD and TOD is more difficult in optical systems. However, recently Chang *et al* [24] have successfully designed mode-locked lasers and chirped-pulse amplifiers operating at several femtoseconds or less by compensating for GVD, TOD and other higher order effects. Moreover, it should be noted that the solutions (42) and (45) may also apply to systems beyond ultrashort pulses as long as the GVD and TOD effects are too small to destroy these solitary waves.

5. Summary

In conclusion, we have obtained multisoliton solutions of the Hirota equation with variable coefficients by the Darboux transformation based on the AKNS technology. As an example, we discuss the evolutional behaviours of two-soliton solution in a soliton control fibre system. The results indicate that one may control the interaction between the pulses by choosing the third-order dispersion parameters d_4 and h appropriately. Because of the invalidation to derive dark soliton by Darboux transformation based on the AKNS technology, we use the extended tanh-function method to obtain the dark soliton and bright soliton solutions of the generalized higher order nonlinear Schrödinger equation, which have more generalized forms than their corresponding solutions in section 2 in this paper and in [8]. Moreover, new bright and dark combined solitary wave, kink solitary wave and M-shaped solitary wave, which cannot exist in the vNLSE, are firstly presented in vHONLSE in this paper by assuming real amplitude instead of assuming imaginary amplitude in [13, 16]. With the absolute of the coefficients A_2 and A_3 gradually decreasing in solution (42), bright and dark combined solitary wave will turn into kink solitary wave. Compared with the results in [9, 11, 13, 16], a control parameter $\alpha_1(z)$ is added in these solutions (17), (18), (28), (29), (42) and (45) in this paper, which make our results more coincident with the real situation. Especially, to these new combined solitary wave (42) and (45), the term proportional to $\alpha_1(z)$ resulting from the group velocity is very important because it determines the group velocity and the phase shift of these new solitary waves. These exact solutions provide powerful theoretical evidence for soliton communication, and can readily be applied to pulse propagation in the femtosecond laser systems.

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